Another example of an infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \leftarrow \text{harmonic series.}$$

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$$
$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 2\frac{1}{12} \cdots$$

slowly increases.

Note that (S_k) is monotone (increasing)

(This will always happen when the series has nonnegative terms, since if $\sum a_n$ is the series, $S_{k+1} = S_k + a_{k+1} \geq S_k$.) $\forall k \in \mathbb{N}$.

Thm. The harmonic series diverges.

Pf. We show that $S_k = \sum_{n=1}^{k} \frac{1}{n}$ is unbounded.

Observe that for $k \in \mathbb{N}$

$$S_{2^k} = \underbrace{\underbrace{\underbrace{1 + \frac{1}{2}}_{k=1} + \left(\frac{1}{3} + \frac{1}{4}\right)}_{k=2} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{k=3}$$

$$+\left(\frac{1}{9}+\dots+\frac{1}{16}\right)+\left(\frac{1}{17}+\dots+\frac{1}{32}\right)+\dots+\left(\frac{1}{2^{k-1}+1}+\dots+\frac{1}{2^k}\right)$$

$$\geq \overbrace{\overbrace{\underbrace{1+\frac{1}{2}}_{k=1}+\left(\underbrace{\frac{1}{4}+\frac{1}{4}}_{2}\right)}^{k=2}+\left(\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{4}\right)}^{k=3}$$

$$+\left(\underbrace{\frac{1}{16}+\dots+\frac{1}{16}}_{8\text{ times}}\right)+\left(\underbrace{\frac{1}{32}+\dots+\frac{1}{32}}_{16\ \#s}\right)+\dots+\left(\underbrace{\frac{1}{2^k}+\dots+\frac{1}{2^k}}_{2^{k-1}\ \#s}\right)$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$+\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2}$$

$$= 1+\frac{k}{2}=\frac{k+2}{2}$$

Scratch

$1+\frac{k}{2}\geq M$

$\frac{k}{2}\geq M-1$

$k\geq 2M-2$

By Archie, $\forall M\in\mathbb{R}$, $\exists k\in\mathbb{N}$ s.t. $k>2M-2$.

But then $k+2>2M \Rightarrow \frac{k+2}{2}>M$

Thus $S_{2^k}>\frac{k+2}{2}>M$.

Therefore, the partial sum sequence (S_n) is unbounded. Therefore, the harmonic series diverges! ☒

Important Lemma:

Lemma: If $\sum a_k$ converges, then $\lim a_n = 0$.

But: the converse is false. (e.g. $\sum \frac{1}{n}$ satisfies $\lim \frac{1}{n} = 0$, but $\sum \frac{1}{n}$ diverges.)

Actually the contrapositive is used more often: If (a_n) is a sequence s.t. either $\lim a_n$ does not exist or $\lim a_n$ exists but is not zero, then the series $\sum a_n$ diverges.

Exercise 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

Let's experiment first before attempting a proof:

$x_1 = 3$ $\quad x_3 = \frac{1}{4-1} = \frac{1}{3}$

$x_2 = 1$ $\quad x_4 = \frac{1}{4 - 1/3} = \frac{1}{11/3}$

$x_4 = \frac{3}{11} < \frac{3}{9} = \frac{1}{3}$

$x_5 = \dfrac{1}{4 - 3/11} = \dfrac{1}{\frac{44-3}{11}} = \dfrac{1}{\frac{41}{11}}$

$= \frac{11}{41} \overset{?}{<} \frac{3}{11}$ looks like it is decreasing.

looks like $0 \leq x_n \leq 3$.

Let's try to use induction

$0 \leq x_1 = 3 \leq 3$. ✓

Sp. $0 \leq x_n \leq 3 \Rightarrow x_{n+1} = \dfrac{1}{4 - x_n} \leq \dfrac{1}{4-3} = \dfrac{1}{1} = 1 \leq 3$

$0 \leq \frac{1}{4} = \frac{1}{4-0} \leq$ will work.

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decreasing. $x_2 = 1$, $x_1 = 3$ so $x_2 \leq x_1$.

Sp. $x_{n+1} \leq x_n$

$x_{n+2} = \dfrac{1}{4 - x_{n+1}} \leq \underbrace{\dfrac{1}{4 - x_n}}_{\text{smaller denom.}} = x_{n+1}$

MCT.

Actual Proof: First, observe that $0 \leq x_1 = 3 \leq 3$.

Suppose for some $n \in \mathbb{N}$, $0 \leq x_n \leq 3$.

Then $0 \leq \frac{1}{4} = \frac{1}{4-0} \leq \frac{1}{4-x_n} \leq \frac{1}{4-3} = 1 < 3$.

Thus $x_{n+1} = \frac{1}{4-x_n}$ satisfies $0 \leq x_{n+1} \leq 3$ also.

Thus, by induction $0 \leq x_n \leq 3$ for all $n \in \mathbb{N}$. ✓

Next, observe that $1 = x_2 \le x_1 = 3$.

Suppose for some $n \in \mathbb{N}$, $x_{n+1} \le x_n$.

Then

$$x_{n+2} = \frac{1}{4 - x_{n+1}} \le \frac{1}{4 - x_n} = x_{n+1}.$$

smaller positive denom.

By induction, $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$.

By MCT, (x_n) converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

From our lemma, $\lim x_{n+1} = \lim x_n$ if $\lim x_n$ exists.

$$x_{n+1} = \frac{1}{4 - x_n}$$

$$\Rightarrow \lim x_{n+1} = \lim \frac{1}{4 - x_n} \; *$$

$$= \frac{1}{4 - \lim x_n} \; **$$

The limit (*) exists because of the ALT and $4 - x_n \ge 1$ (so that $\lim 4 - x_n \ge 1$), and the result is ($\star$) by ALT.

Let $L = \lim x_n = \lim x_{n+1}$.

$\Rightarrow \quad L = \frac{1}{4-L} \Rightarrow L(4-L) = 1$

$L(4-L) - 1 = 0$

$4L - L^2 - 1 = 0 \Rightarrow L^2 - 4L + 1 = 0$

$\Rightarrow \quad L = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2}$ $\quad 2\sqrt{3}$

$= 2 \pm \sqrt{3}.$

$0 \le x_n \le 3 \Rightarrow$ by OLT

that $0 \le \lim x_n = L \le 3$

$0 < 2-\sqrt{3} < 3 < 2+\sqrt{3}$

$\Rightarrow \boxed{L = 2-\sqrt{3}}$. $\blacksquare$

Exercise 2.4.3. (a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

(b) $x_1 = \sqrt{2}$

$x_{n+1} = \sqrt{2x_n}$

Thinking: looks like it is increasing.

Induct: $x_2 = \sqrt{2\sqrt{2}} \overset{?}{\geq} \sqrt{2} = x_1$

$\sqrt{2} \geq 1$

$2\sqrt{2} \geq 2$

$\sqrt{2\sqrt{2}} \geq \sqrt{2}$. ✓

Suppose $x_{n+1} \geq x_n$ for some n.

$x_{n+2} = \sqrt{2x_{n+1}} \geq \sqrt{2x_n} = x_{n+1}$ ✓

induct $\Rightarrow$ increasing.

Bounded:

$\sqrt{2} \leq x_n$ ✓.

Guess $x_n \leq 1000$

$x_1 = \sqrt{2} \leq 1000$. ✓

Suppose $x_n \leq 1000$.

Then $x_{n+1} = \sqrt{2x_n} \leq \sqrt{2 \cdot 1000} = \sqrt{2000} < 1000$. ✓

MCT $\rightarrow$. ✓

Pf. Observe $x_2 = \sqrt{2\sqrt{2}} \geq \sqrt{2 \cdot 1} = x_1$.

Sp. $x_{n+1} \geq x_n$ for some $n \in \mathbb{N}$.

Then $x_{n+2} = \sqrt{2x_{n+1}} \geq \sqrt{2x_n} = x_{n+1}$.

By induction, $x_{n+1} \geq x_n \ \forall n \in \mathbb{N}$.

Next, observe $\sqrt{2} \leq x_1 \leq 1000$

Sp. $\sqrt{2} \leq x_n \leq 1000$.

Then $\sqrt{2} \leq x_n \leq x_{n+1} = \sqrt{2x_n} \leq \sqrt{2 \cdot 1000}$

$= \sqrt{2000} < 1000$.

Thus $\sqrt{2} \leq x_n \leq 1000 \ \forall n \in \mathbb{N}$.

By MCT, (x_n) converges.

How do we find the limit?

Since $\lim x_n$ exists, let

$L = \lim x_n = \lim x_{n+1}$ by lemma.

$\Rightarrow$ By recursive formula

$$L = \sqrt{2L}$$ by ALT

($L \geq \sqrt{2} > 0$) by OLT.

$$\Rightarrow L^2 = 2L$$

$$\Rightarrow L^2 - 2L = 0$$

$$L(L-2) = 0$$

$\Rightarrow L=2$ or $L=0$.

By OLT $L \geq \sqrt{2} \Rightarrow \boxed{L=2}$.